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Comment on "Casimir force in compact non-commutative extra dimensions and radius stabilization"

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ABSTRACT: We call attention to a series of mistakes in a paper by S. Nam recently published in this journal (*J. High Energy Phys.* **10** (2000) 044).

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In a recent paper [1], S. Nam discussed a possible mechanism for the stabilization of the compact dimensions in a Kaluza-Klein scenario. He considered the hypothesis of the geometry in the compact dimensions being non-commutative, i.e. the coordinates in these dimensions obeying the commutation relations $[x^\mu, x^\nu] = i\,\theta^{\mu\nu}$, where $\theta^{\mu\nu}$ is an antisymmetric matrix. This gives rise to the uncertainty relations $\Delta x^\mu\,\Delta x^\nu \geq \frac{1}{2}\,|\theta^{\mu\nu}|$, implying the existence of a minimum area in the (μ,ν)-plane, which would prevent the collapse of the compact dimensions.

In order to test this idea, Nam computed the one-loop Casimir energy for the massless ϕ^3 theory and the U(1) gauge theory defined on $\mathbb{R}^{1,d} \times \mathbb{T}^2_\theta$, where $\mathbb{R}^{1,d}$ is an ordinary $(1+d)$-dimensional Minkowski space-time and $\mathbb{T}^2_\theta$ is a non-commutative two-torus whose coordinates satisfy $[y^1, y^2] = i\,\theta$ and $0 \leq y^1, y^2 \leq 2\pi R$. He arrived at the conclusion that such a mechanism is not effective in the scalar case, but it does work in the vector case for $d > 5$. Unfortunately he made a series of mistakes which render his conclusions untenable. The purpose of this Comment is to discuss those mistakes.

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First mistake. In ref. [2] Gomis et al. computed the one-loop correction to the Kaluza-Klein spectrum for the ϕ^3 theory and the U(1) gauge theory defined on $\mathbb{R}^{1,3} \times \mathbb{T}^2_\theta$. Nam tacitly assumed that this correction is independent of the dimension d — a wrong assumption, as Huang has shown [3]. As a consequence, Nam's results are not valid for $d \neq 3$.

Second mistake. Even for $d = 3$, Nam's calculation of the Casimir energy density is incorrect. To begin with, the sum in eq. (13) of his paper is incomplete; it should run over all pairs of integers (n_1, n_2) in the range $-\infty < n_1, n_2 < \infty$ (with the exception of the pair $n_1 = n_2 = 0$ [2]), instead of $n_1, n_2 \geq 1$. Therefore, the function v_2 in ref. [1, eqs. (17)–(18)] should be replaced by

$$\tilde{v}_2(z) \equiv \sum_{m,n=-\infty}^{\infty}{}' \left(m^2 + n^2\right)^{-z}\,, \tag{1}$$

the prime meaning that the term with $m = n = 0$ is omitted.

The sum in eq. (1) is convergent for $\operatorname{Re} z > 1$, so that [1, eq. (17)] is well defined for $d < -3$. Before we can set $d = 3$, we have to analytically continue $\tilde{v}_2(z)$ to $\operatorname{Re} z < 1$. To our purposes it is enough to use the reflection formula [4]

$$\pi^{-z}\,\Gamma(z)\,\tilde{v}_2(z) = \pi^{z-1}\,\Gamma(1-z)\,\tilde{v}_2(1-z)\,, \tag{2}$$

which allows us to rewrite [1, eq. (17)] as $[\alpha \equiv \lambda^2/(4\pi)^3\theta^2]$

$$\begin{aligned} u(R) = -\frac{\mu^{3-d}}{2^{d+2}\,\pi^{(d+1)/2}} \Bigg[& \pi^{-d-2}\,\Gamma\left(\frac{d+3}{2}\right)\tilde{v}_2\left(\frac{d+3}{2}\right) R^{-d-1} + \\ & + \alpha\,\frac{2\pi^{-d+2}}{1-d}\,\Gamma\left(\frac{d-1}{2}\right)\tilde{v}_2\left(\frac{d-1}{2}\right) R^{-d+3} + \ldots \Bigg]\,, \end{aligned} \tag{3}$$

which is now well defined for $d > 3$. (We have introduced an arbitrary mass scale μ to keep u a three-dimensional energy density.) Expanding u in powers of $\epsilon = d - 3$, we obtain

$$u(R) = -\frac{\tilde{v}_2(3)}{16\pi^7 R^4} + \frac{\alpha}{16\pi^2}\left(\frac{1}{\epsilon} - \ln(\mu R) + \text{const}\right) + O(\epsilon)\,. \tag{4}$$

(To arrive at this expression, we have used the identity $\tilde{v}_2(z) = 4\,\zeta(z)\,\beta(z)$ [4], where $\zeta(z)$ is the Riemann zeta function and $\beta(z) \equiv \sum_{n=0}^{\infty}(-1)^n\,(2n+1)^{-z}$.)

The radiative correction to u diverges when $d \to 3$. However, the divergent contribution does not depend on R, and so it can be subtracted from u, as it has no effect on the Casimir force. In order to fix the finite part of the correction, we impose that the renormalized Casimir energy density vanishes at some arbitrary radius $R = R_*$; then

$$u(R) = -\frac{\tilde{v}_2(3)}{16\pi^7}\left(\frac{1}{R^4} - \frac{1}{R_*^4}\right) - \frac{\alpha}{16\pi^2}\ln\left(\frac{R}{R_*}\right). \tag{5}$$

In contrast with the result found by Nam in the case $d = 3$, the above function has an extremum, located at

$$R = \left(\frac{4\,\tilde{v}_2(3)}{\pi^5\,\alpha}\right)^{1/4}. \tag{6}$$

This, however, does not imply the stabilization of the compact dimensions, since this extremum is a maximum of u. This leads us to Nam's

Third mistake. Nam obtained for the U(1) gauge theory on $\mathbb{R}^{1,d} \times \mathbb{T}^2_\theta$ a Casimir energy density of the form

$$u(R) = -\frac{a}{R^{d+1}} + \frac{b}{R^{d-5}}\,, \tag{7}$$

with a and b functions of d. He then argued that radius stabilization occurs for $d > 5$, for in this case the ratio b/a is positive and there is a radius R_0 such that $u'(R_0) = 0$. However, he overlooked the fact that $u''(R_0) < 0$, so that the equilibrium is *unstable*.

Fourth mistake. Nam did not take into account all $O(\lambda^2)$ corrections to u. Indeed, the Kaluza-Klein spectrum he used as the starting point of his calculation takes into account only the non-planar contribution to the one-loop self-energy [2]. However, the inclusion of the planar contribution is not enough to correct the result. To show why, we start by noting that the Casimir energy density can also be expressed as[1] [5]

$$\epsilon = \langle 0_R|\,T_{00}\,|0_R\rangle\,, \tag{8}$$

[1]While u is a three-dimensional energy density, i.e., $u = E/\mathrm{Vol}(\mathbb{R}^3)$, ϵ is a five-dimensional density: $\epsilon = E/\mathrm{Vol}(\mathbb{R}^3 \times \mathbb{T}^2_\theta) = u/(2\pi R)^2$.

where T_{00} is the 00-component of the energy-momentum tensor,

$$T_{00} = \frac{1}{2}\,(\partial_0\phi)^2 + \frac{1}{2}\,(\nabla\phi)^2 + \frac{\lambda}{3!}\,\phi\star\phi\star\phi\,, \tag{9}$$

and $|0_R\rangle$ is the vacuum state corresponding to a compactification radius R. (Strictly speaking, such a state does not exist in the present case, as the Hamiltonian H is unbounded from below. In spite of this, one can still give a well defined meaning to eq. (8) by interpreting $|0_R\rangle$ as the state which minimizes $\langle H\rangle$ under the constraint $\langle\phi(x)\rangle = 0$.)

Moving the derivatives outside the brackets, one can rewrite eq. (8) as

$$\epsilon(x) = \lim_{x'\to x}\frac{1}{2}\,(\partial_0\partial_0' + \partial_i\partial_i')\,G^{(2)}(x,x') + \frac{\lambda}{3!}\,G^{(3)}_\star(x,x,x)\,, \tag{10}$$

where $G^{(2)}(x,x') \equiv \langle\phi(x)\phi(x')\rangle$ is the (renormalized) connected two-point Green's function, and $G^{(3)}_\star(x_1,x_2,x_3) \equiv \langle\phi(x_1)\star\phi(x_2)\star\phi(x_3)\rangle$. The former satisfies the Schwinger-Dyson equation,

$$-\partial^2 G^{(2)}(x,x') - \int dy\,\Sigma(x,y)\,G^{(2)}(y,x') = i\,\delta(x-x')\,, \tag{11}$$

where Σ, the self-energy, is (i times) the sum of all 1PI diagrams with two (amputated) external lines. Using eq. (11) to eliminate the spatial derivatives in eq. (10), and noting that: (i) due to translation invariance, $\partial_\mu' G^{(2)}(x,x') = -\partial_\mu G^{(2)}(x,x')$, and (ii) $\lim_{x'\to x}\delta(x-x') = 0$, we arrive at

$$\epsilon(x) = \lim_{x'\to x}\,\partial_0\partial_0'\,G^{(2)}(x,x') - \frac{1}{2}\int dy\,\Sigma(x,y)\,G^{(2)}(y,x) + \frac{\lambda}{3!}\,G^{(3)}_\star(x,x,x)\,. \tag{12}$$

Using the spectral representation of $G^{(2)}$, one can recast the first term on the r.h.s. of eq. (12) as a sum of zero-point energies. Nam's calculation is equivalent to the computation of that term with $G^{(2)}$ approximated by

$$G^{(2)}(x,x') \approx i\left\langle x\left|\frac{1}{-\partial^2 - \Sigma^1_{\mathrm{NP}}}\right|x'\right\rangle\,, \tag{13}$$

where Σ^1_{NP} is the non-planar piece of the one-loop self-energy. Hence, Nam's calculation includes radiative corrections at $O(\lambda^2)$. However, besides neglecting the planar piece of Σ, Nam overlooked the last two terms on the r.h.s. of eq. (12), which also contain an $O(\lambda^2)$ contribution to the Casimir energy.

In order to estimate the effect of those terms, let us consider, for the sake of simplicity, the commutative $\frac{\lambda}{3!}\,\phi^3$ theory. Then, in the lowest non-trivial order in λ the self energy is given by

$$\Sigma(x,y) = -\frac{i\lambda^2}{2}\,[G_0(x,y)]^2 + O(\lambda^4)\,, \tag{14}$$

where G_0 is the free two-point Green's function, and the three-point Green's function reads

$$G^{(3)}(x,x,x) = -i\lambda \int dy \, [G_0(x,y)]^3 + O(\lambda^3) \, . \tag{15}$$

The lowest order radiative correction to the Casimir energy density is thus given by

$$\epsilon^{(2)} = \epsilon^{(2)}_{\text{Nam}} + \frac{i\lambda^2}{12} \int dy \, [G_0(x,y)]^3 \, , \tag{16}$$

where $\epsilon^{(2)}_{\text{Nam}}$ is the result one obtains (in the commutative theory) by following Nam's approach, i.e. by summing the one-loop zero-point energies.

The integral in eq. (16) can be evaluated indirectly by noting that the Casimir energy density [or, more precisely, $V_{\text{eff}}(\langle\phi\rangle = 0)$] is also given by

$$\epsilon = \lim_{V,T\to\infty} \frac{i}{VT} \ln Z \, , \tag{17}$$

where

$$Z = \int [\mathcal{D}\phi]_{\langle\phi\rangle=0} \exp\left\{ i \int dx \left[\frac{1}{2} (\partial\phi)^2 - \frac{\lambda}{3!} \phi^3 \right] \right\} . \tag{18}$$

(To satisfy the constraint $\langle\phi\rangle = 0$, one simply ignores any 1PI one-point diagrams in the perturbative expansion of Z [6].) In this approach, the second order term in the perturbative expansion of ϵ is given by

$$\epsilon^{(2)} = -\frac{i\lambda^2}{12} \int dy \, [G_0(x,y)]^3 \, . \tag{19}$$

Comparing with eq. (16), we obtain

$$\epsilon^{(2)}_{\text{Nam}} = -\frac{i\lambda^2}{6} \int dy \, [G_0(x,y)]^3 = 2 \, \epsilon^{(2)}, \tag{20}$$

i.e. the lowest order radiative correction to ϵ computed à la Nam is twice the correct value. We expect a similar correction in the non-commutative theory. However, this has to be checked by an explicit computation.

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